1 THE CONCEPT OF SAMPLE SPACE IN PROBABILITY

INTRODUCTION

A sample space is the set of all possible outcomes of an experiment. It is used to calculate the probabilities of an experiment in both mathematics and real–life situations. In this chapter, students are exposed to probabilistic activities and asked to calculate the sample spaces of given situations. In addition, the chapter explores probability and hence sample space usage in science. The lessons are mainly activity based and student centered.

OBJECTIVES

Mathematics

– Students are expected to construct sample spaces using lists.

Science

– Students are expected to recognize that inherited traits of individuals are governed in the genetic material found in the genes within chromosomes in the nucleus.

Technology

– Students are expected to use appropriate software components which are suitable for the task.

– Students are expected to use technology applications to facilitate their work.

MATERIALS

– Computer for each group

– Worksheet 1

– Worksheet 2

– LCD projector to show videos

STUDENT INTRODUCTION

Have you ever gambled on horse races or played the lottery? Have you ever gained money from such activities? What are your chances of winning in those activities? There are so many people who engage in those activities every year all over the world. Any situation that involves chance or probability is called an experiment. The set of all possible outcomes in such an activity is called a sample space. As you can easily predict, if the number of outcomes increases, then the probability of winning in those activities decreases. In other words, if the number of outcomes in a sample space is high, then the probability of guessing the winning result is low. The concept of sample space is the key component in finding the likelihood of a probability experiment. In the later parts of probability study, in order to determine the probability we need to calculate the sample space. Otherwise, the probability of an event cannot be determined. In the subsequent lessons, we will go into the concept of sample space, what it means, and how to calculate it.

ENGAGEMENT

Before starting this activity, the teacher divides the students into teams of two to perform an activity. Then, he or she gives each group two urns. The first urn contains ten red and five blue marbles, and the second urn contains ten red and twenty blue marbles. Then, the teacher asks the students to pick five marbles from each urn and record the results. One student picks the marbles and the other student records the results. After students perform all picking and recording, the teacher asks in which urn it is more likely to pick a blue marble. Then, the teacher asks the students to discuss the question by linking it to what they have recorded. This part simply intends to develop a brief understanding of the concept of sample space and to encourage students to think a little bit about the sample space. Together with a teacher–led discussion, this part takes approximately one hour.

EXPLORATION

In this stage, the teacher helps students to explore the meaning of sample space. The teacher divides the students into teams of four. Each group performs the activity independently. Before starting the activity, the teacher

provides some introductory information and asks some thought–provoking questions. For example, he or she should ask questions such as which cell has the highest occurrence rate. Such questions might direct students to think more sensitively and appropriately for developing an understanding of the logic of probability. Overall, the activity consists of two parts. In the first part, students have six prisoners whom they are trying to release. Students roll two dice, calculate the difference, and release the prisoner if that number is one. In addition, questions are stated in the second part that are important, because they include guiding questions that lead students through an initial exploration of the concept of sample space. For this reason, the teacher should not let students skip any questions.

Release the Prisoners Game. Worksheet 1

In this first part of the game, you have six prisoners, whom you want to release.

Game Rules

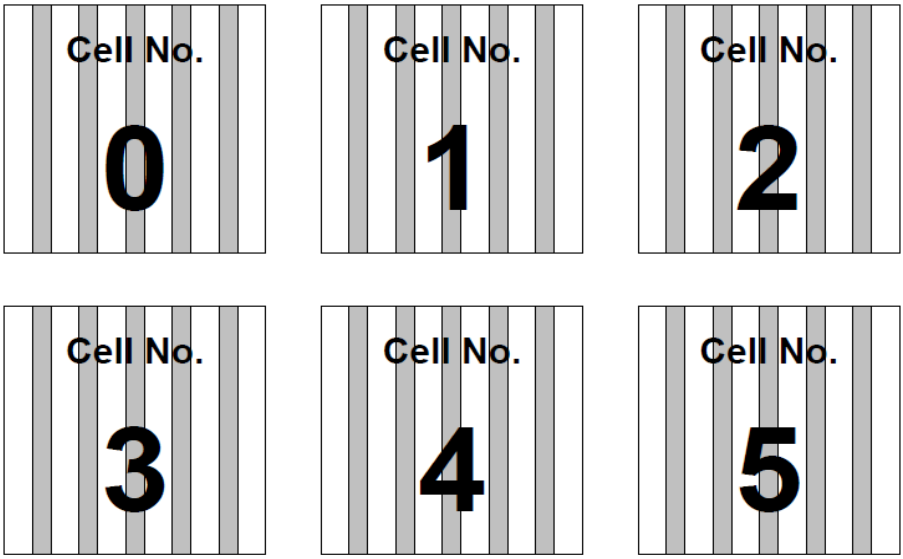
– You have six prisoners and six cells; you are free to choose how to place the prisoners into the cells.

– You are not obligated to place the six prisoners into different cells. You can place all of them into one cell, you can distribute all of them into different cells, and so on.

– You have two dice. Roll them, calculate the difference, and release one of your prisoners, if that number is one.

– There are two other people performing the same activity. The first to release all of his or her prisoners in the given time interval is the winner.

– Good luck!



Release the Prisoners Game. Worksheet 2

In this part of the game, you have five questions to answer. Answer them carefully.

1. Is the game fair? What do you think? Explain your answer in a few sentences.

2. Who is the most successful person in your group? Why is that person the best? Is he or she very lucky, or is there another reason behind his or her success?

3. Could choosing the numbers that have the highest frequency or occurrence rate be a reason for the best player's success?

4. If you were to replay the game, which numbers would you choose in both parts?

5. Can you make a list of all the outcomes that can be drawn in both cases? What do we call it?

After students perform the activity, the teacher starts a discussion about the students' findings and then asks those who did not win about their failure. In this way, the teacher gets the students to query themselves to give them a chance to learn from their failure. This provides a step–by–step exploration of the notion of sample space.

EXPLANATION

First, the teacher asks students to make a tabular representation of the results of rolling two dice. In so doing, students have a chance to look at the entire sample space and see the mistakes that they made in the first part of the activity. The teacher initially asks the students to find the results of rolling two coins and then asks them to calculate the difference of the numbers in the sample space, as was required in the first part of the activity.

Table 1

	1	**2**	**3**	**4**	**5**	**6**
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

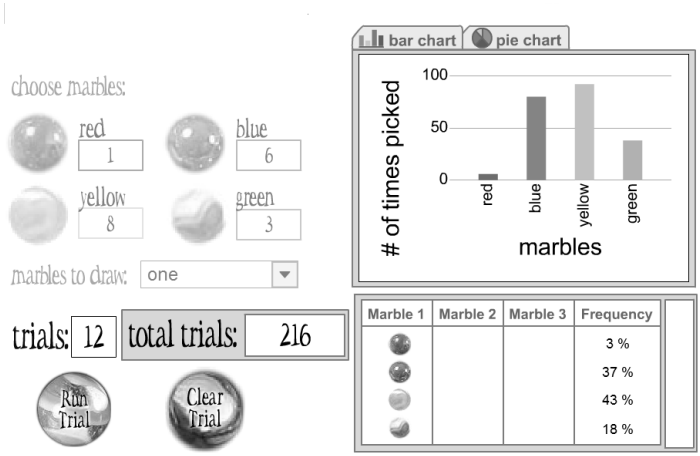
Table 2

	1	**2**	**3**	**4**	**5**	**6**
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

In the table, the teacher shows students that one is the most frequently occurring number, with ten occurrences in the sample space. Two is the second most frequent number, with eight occurrences, three and zero are third, with six occurrences, four is fourth, with four occurrences, and five is the least frequent, with two occurrences. If the students had thought to make

this tabularization or at least to imagine the picture beforehand, they would have had better results.

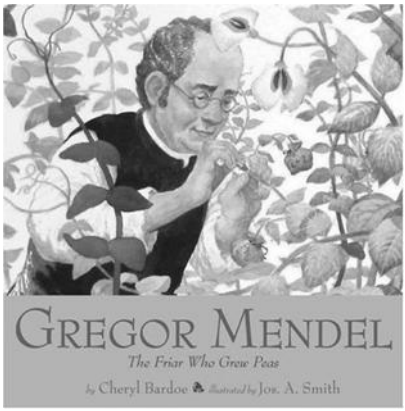
Now they understand better the meaning of sample space. Then, the teacher asks the students to visit the web page located at http://sciencenetlinks.com/media/filer/2011/10/13/marblemania.swf to open the Marble Mania! activity. Directing this activity is very easy.



Students can increase the number of marbles and, hence, change the sample space. The teacher asks the students to determine the number of marbles and then to start the trial. There are two options, such as a bar chart and a pie chart. Students should be allowed to determine the type of chart. For example, let us perform a trial: choose one red, six blue, eight yellow, and three green marbles, so there are 18 marbles in our sample space. The bulk of the marbles are blue and yellow, so we can presume that more than half of the marbles drawn from the file would be blue or yellow. At this point, students should derive that conclusion. If they do not, the teacher reviews the previous activity. Then, the teacher shows a video about Mendel's laws on the web page located at

http://www.bbc.co.uk/schools/gcsebitesize/science/add_aqa/inheritance/genetic_variationact.shtml

The video is about the genes in the human body that come from the father and mother. Having brown eyes is dominant in the sample space. The teacher briefly goes over the details.



Gregor Johann Mendel was an Austrian scientist and cleric who achieved posthumous fame as the founder of the new science of genetics. He was a gardener who worked with peas. Mendel studied the alleles that come from chromosomes.

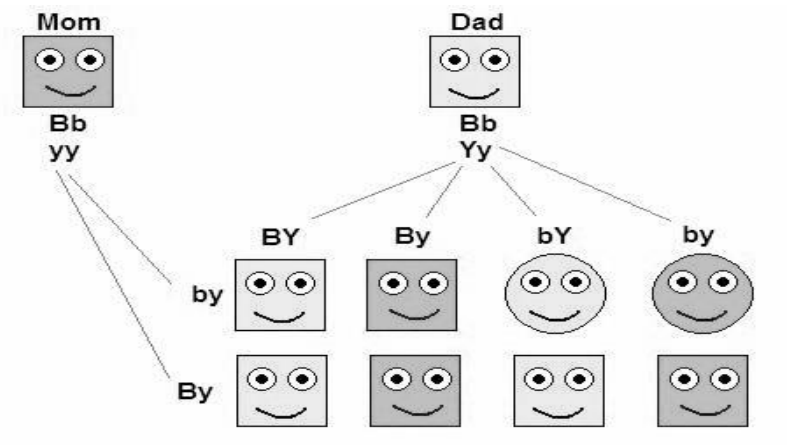


Figure 1. Crossing the alleles coming from the parents

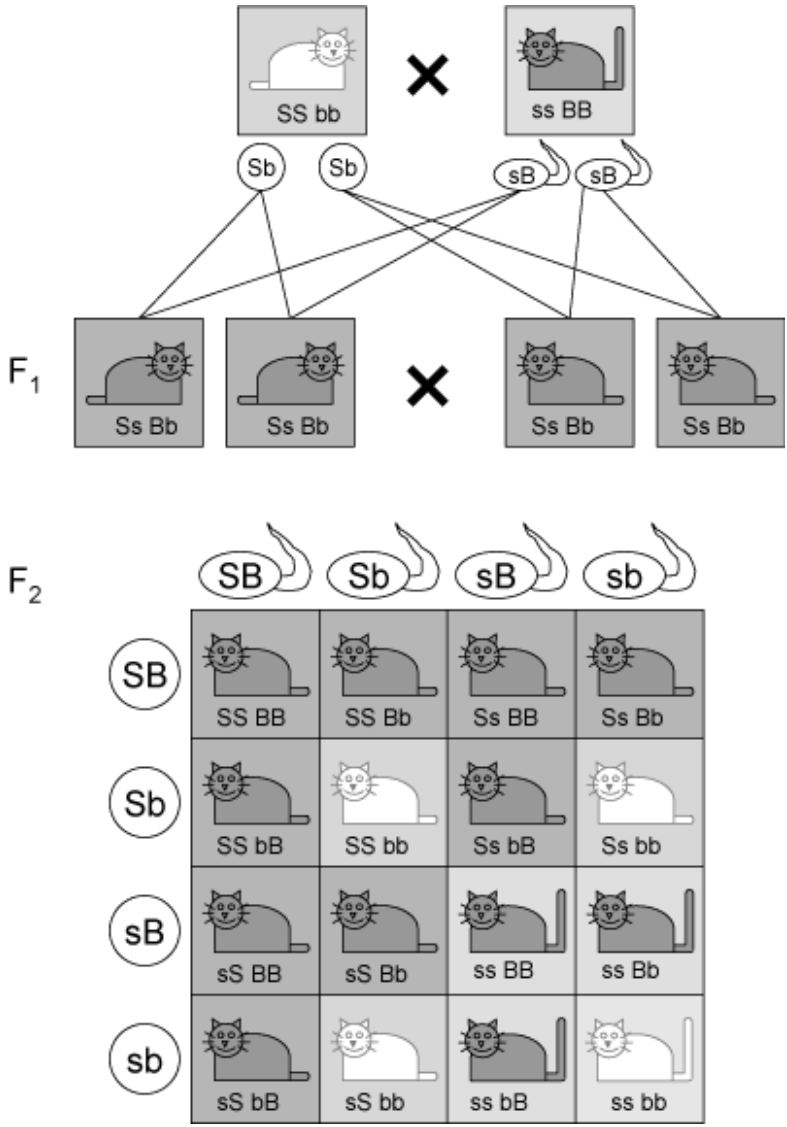


Figure 2. An example of crossing in cats

Using tree diagrams to find the sample space: The teacher asks the students for the sample space of flipping two coins. The students have already seen the sample space, so getting the correct results should not be difficult. Then, the teacher shows the tree diagram of tossing two coins.

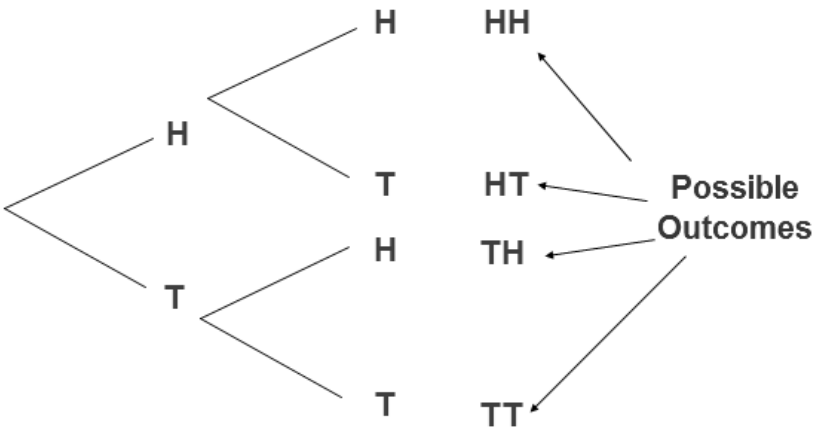


Figure 3. Tree diagram of flipping two coins

Next, the teacher explains the construction of tree diagrams using different examples, as follows:

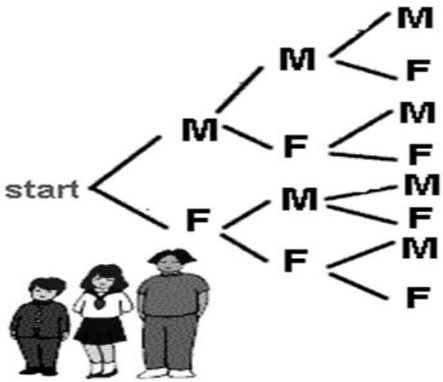


Figure 4. Children in a family

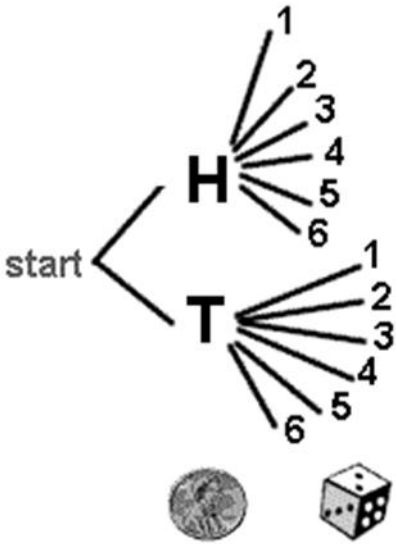


Figure 5. Tossing a coin and rolling a die

EXTENTION

The teacher opens a stick or switch activity from Utah State University's library of virtual manipulatives. This is an entertaining activity that tests students' knowledge of the concept of sample space. In the activity, there are three doors, only one of which is the winning one. Here is a picture of what it looks like:

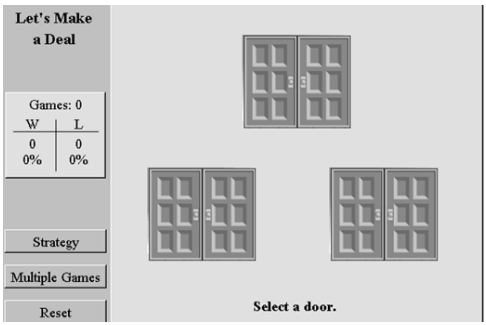


Figure 6. Stick or switch activity

The strategy of the game is simple. There are three doors, and we want to find the winning one. However, two trials are offered to find the winning door. First, we choose a door. Then the program opens another door, so we have the two options of either sticking with the same door or switching the door. The teacher asks students to find the strategy that produces the most productive outcome.

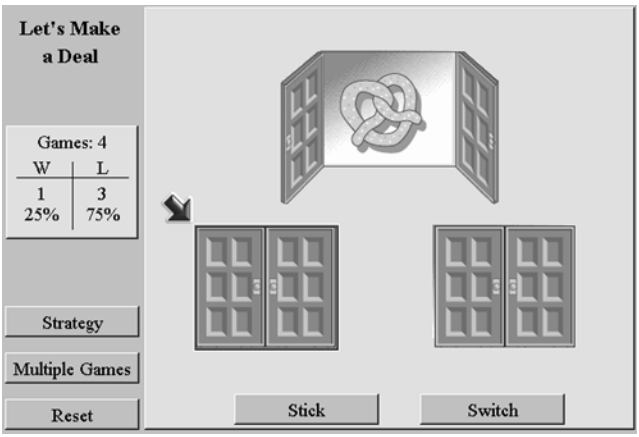


Figure 7. Stick or switch activity

The picture shows how the activity looks after one trial. The program shows two doors, one of which is a losing door, and offers two options, stick or switch.

Hint: Use a tree diagram to list all of the outcomes to determine what strategy is most useful for winning most of the time.

EVALUATION

True or False

Circle the correct answer.

1. A sample space is a set of all possible outcomes. [True/False]

2. The sample space of a tossing–a–coin activity consists of six elements. [True/False]

3. In a tossing–two–coins activity, the sample space contains four elements. [True/False]

4. If we remove a possible outcome from the set of all possible outcomes, the sample space does not change. [True/False]

5. With two four–sided dice, the sample space of the experiment of tossing two dice consists of eight elements. [True/False]

6. In a rolling–two–dice–experiment, the sample space consists of 36

elements. [True/False]

7. There are 3 blue, 5 purple, 6 green, and 12 white marbles in an urn. The sample space of drawing a marble from the urn has 26 outcomes. [True/False]

8. When we roll a die and toss a coin, then the sample space consists of 12 outcomes. [True/False]

9. The sample space of the tossing–four–coins experiment contains 16 outcomes. [True/False]

10. A tree diagram is a tool that is used to show all possible outcomes of an activity. [True/False]

Answer Sheet

1) T, 2) F, 3) T, 4) F, 5) F, 6) T, 7) T, 8) T, 9) T, 10) T

Short Answer

1. What is the term that means a situation involving chance or probability and producing results called outcomes?

2. What do we call the set {1, 2, 3, 4, 5, 6} in a coin tossing experiment?

3. What are the symbols that we use to show a sample space?

4. What is the name of a circle with an arrow in the center that turns along the circular path?

5. What would be the number of outcomes in a tossing–a–coin–and–rolling–a–die experiment?

Answer Sheet

1. Experiment

2. Sample Space

3. S, Ω, or U

4. Spinner

5. 12

Multiple–Choice Questions

1. What is the sample space for choosing an odd number from one to ten at random?

a. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

b. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

c. {1, 3, 5, 7, 9}

d. {2, 4, 6, 8, 10}

2. What is the sample space for choosing a prime number less than 13 at random?

a. {2, 3, 5, 7, 11, 13}

b. 2, 3, 5, 7, 11, 13

c. {2, 3, 5, 7, 9, 11, 13}

d. {1, 3, 5, 7, 9}

3. What is the sample space for choosing one jelly marble at random from a jar containing five red, seven blue, and two green jelly marbles?

a. {5, 7, 2}

b. {5 red, 7 blue, 2 green}

c. {red, blue, green}

d. {5 red, 7 blue}

4. What is the sample space for choosing one consonant at random from the word 'Space'?

a. {S, P, C}

b. {S, P, A, C, E}

c. {A, E}

d. {S, P, C, E}

5. What is the number of possible outcomes from drawing a card from a deck?

a. 52

b. 51

c. 13

d. 15

6. What is the sample space if two coins are tossed?

a. {T, H}

b. {T, T}

c. {H, H}

d. {(T, T), (T, H), (H, T), (H, H)}

7. What is the sample space if a coin is rolled?

a. {1, 2, 3, 4, 5, 6}

b. {1, 3, 4, 5, 6}

c. {1, 2, 3, 5, 6}

d. {0, 1, 2, 3, 4, 5}

8. In a jar there are three blue, two red, and six purple marbles. A marble is drawn from the jar, and we know the color of the marble is not blue. In this sense, what would be the new sample space for this experiment?

a. {3 blue, 2 red, and 6 purple}

b. {2 red and 6 purple}

c. {3 blue, 2 red}

d. {3 blue and 6 purple}

9. If the sample space of an experiment is {(T, T), (T, H), (H, T), (H, H)}, then what could be the experiment?

a. Tossing two coins

b. Tossing a coin and rolling a die

c. Rolling a die and drawing a card from a deck

d Tossing two coins

10. A box contains tickets numbered from one to 15. John picks a ticket and says the number on the ticket is not odd. Then, according to John's statement, what would be the sample space?

a. {1, 2, 3, 4, 5, 6, 7}

b. {2, 3, 5, 7, 11, 13}

c. {2, 4, 6, 8, 10, 12, 14}

d. {1, 3, 5, 7, 9, 11, 13, 15}

11. If the sample space of an experiment is {5 red, 7 blue, 2 green}, then what could be the experiment?

a. Drawing a card from a deck

b. Picking a marble from jar that contains five red, seven blue, and two green marbles

c. Drawing a ticket from a container that contains seven blue and seven red tickets

d. Rolling a die and tossing a coin

12. What is the sample space of the experiment if we want to pick a vowel from the letters of the word 'Turkey'?

a. {T, R, K}

b. {U, E}

c. {T, R, K, Y}

d. {T, U, R, K}

13. What is the number of outcomes in the experiment if we want to draw a vowel from the word 'Mississippi'?

a. 1

b. 11

c. 4

d. 7

14. What is the sample space if two coins are tossed and we know the faces come up the same?

a. {(T, T), (H, H)}

b. {(H, H)}

c. {(T, T), (H, H), (H, T)}

d. {(T, T), (T, H), (H, T), (H, H)}

15. What is the sample space of tossing two coins if the sum of the numbers that come up is six?

a. {(1, 4), (1, 5), (2, 4), (1, 6)}

b. {(1, 4), (1, 4), (1, 4), (1, 4)}

c. {(2, 4), (1, 5), (3, 3), (4, 2), (5, 1)}

d. {(3, 4), (3, 3)}

Answer Sheet

1) B, 2) C, 3) B, 4) A, 5) A, 6) D, 7) A, 8) B, 9) D, 10) C, 11) B, 12) C, 13) C, 14) A, 15) C

ALTERNATIVE ASSESSMENT

Task 1: The Horse Race Problem

In a horse race, there are five horses, named Hansen, Gemologist, Prospective, Liaison, and Padrino. According to the rules of the race, whoever guesses the first three horses correctly wins $500. You can make only one bet on a ticket. For example, you could select Hansen as the first horse, Gemologist as the second, and Padrino as the third. A ticket costs $10. You want to calculate all possible results of the race and bet on each one to be sure of winning the race to get the $500. How many choices must you make to be sure of winning the race? How much money will it cost? Is it rational to invest that much money in the race? Solve the problem and explain your ideas explicitly.

Hint: Make a tree diagram to calculate all possible outcomes of the horse race.

Rubric 1: The Horse Race Problem

	Understanding the problem	**Planning a solution**	**Getting an answer**
4–3	Students fully understand the problem and linked it to the concept of sample space.	Students use the rules for constructing tree diagrams smoothly and properly.	A tree diagram of all of the possibilities of the race is drawn without error, the number of outcomes is counted, the total price is calculated, and the correct inference about investing the money is drawn.
2–1	The problem is misunderstood or lacks some features relating to the sample space concept.	Students fail to use the rules that are mentioned in class to construct tree diagrams efficiently.	An incomplete tree diagram results in the sample space of the race not being finished, there are errors in counting the number of elements in the sample space, or faulty inferences are drawn about investing $500 in the game.
0	The problem is totally misunderstood.	No strategy or an incorrect strategy is followed to handle the problem.	There is no tree diagram or an incorrect one, the number of outcomes in the sample space is calculated incorrectly, or an incorrect inference or no inference is drawn.

Task 2: Names of the Children in a Family

Suppose a family has four children, all of them boys, named Jack, Adam, Adriano, and George. The brothers are born in successive years. What are all possible choices for the names of the children from the eldest to the youngest? For example, here is one possible order: {Jack, Adam, Adriano, George}. In this order, the eldest boy is named Jack, and the youngest is named George. Likewise, the second child is Adam, and the third is Adriano. Make a list of all possible outcomes, and state the method and the strategy you use to calculate all possible outcomes.

Rubric 2: Names of the Children in a Family

	Understanding the problem	**Planning a solution**	**Getting an answer**
4–3	The children problem is totally understood, and students refer it to the sample space concept.	Students use an efficient plan to solve the problem, and make efficient use of the rules mentioned in class to draw a tree diagram.	Students correctly calculate all possible outcomes with reference to the ages of the children in the problem.
2–1	The problem is only partially understood or is misunderstood, or the link between the question and the sample space is not established.	Students do not use an efficient strategy to calculate the number of outcomes.	Students miss some of the required calculations, or some of the boys' names are wrong.
0	Students completely misunderstand the problem.	Examples or methods given in the lecture are not used to solve the problem, or no strategy is followed.	The answer is missing or totally wrong.

Task 3: Dice Differences Activity

Suppose two friends play a game. Let us call them player A and player B. In the game, they roll two dice in turn and calculate the difference between the two dice. If the result is equal to zero, one, or two, then player A makes a mark on the table, and if the result is three, four, or five, then player B makes a mark on the table. The one who reaches ten marks first will be the winner. Is the game fair? What do you think? Answer this question by listing all possible outcomes of rolling two dice in a table. Then, make another table that shows the differences in numbers of each possible solution. For example, the cell (6, 1) corresponds to the cell 5 in the second table. After you construct the second table, paint the differences in which player A draws the tic. Construct your tables on paper, and then explain your ideas about the fairness of the activity with rational ideas that are linked to the percentages of the differences that each player has for making a mark.

Rubric 3: Dice Differences Activity

	Understanding the problem	**Planning a solution**	**Getting an answer**
4–3	Students understand the problem completely, and students see the unfairness in the activity.	Students apply the suggested method to solve the question step by step.	Tables are completed, the sample space of rolling two dice is correctly defined, and the activity's unfairness is supported with valid reasoning.
2–1	Part of the problem is misunderstood or misinterpreted.	Students try to follow the method suggested for the problem, but deviate at some points.	Errors occur in calculating the sample space, mistakes occur in constructing the tables, or the unfairness of the activity is explored but not explained elegantly or rationally.
0	Students completely misunderstand the problem.	The road map suggested in the question is not followed, or there is no strategy or plan to solve the question.	No sample space is given, the tables are missing, or nearly all of the items are wrong.

Rubric 4 (Extension Activity)

	Developing a strategy to win most of the time.
4	The sample space of the given chance game is completely explained, using that strategy students reach the result changing to what would be the most beneficial for winning most of the time, and ideas are supported with valid reasoning.
3	The sample space is given correctly, students reach the result changing to what would be the most beneficial for winning most of the time, but ideas are not supported with valid reasoning.
2	The sample space is correctly found, but students cannot conclude that changing would be most beneficial for winning most of the time, or ideas are not supported with valid reasoning.
1	Calculation of the sample space includes errors, students cannot conclude that changing would be most beneficial for winning most of the time, or ideas are not supported with valid reasoning.
0	Very little or no effort is exerted even to find the sample space.

AUTHOR

Melih Akar

Primary Education Department

Boğaziçi University